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Stability of large-amplitude geostrophic flows localized in a thin layer

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In this paper the dynamics of geostrophic flows localized in a thin layer of continuously stratified fluid, which overrides a thick homogeneous layer are studied. The displacement of isopycnal surfaces is assumed large; the β-effect is strong, i.e.

$$(R_0/R_e)\cot\theta \gtrsim \epsilon,$$

where ϵ is the Rossby number, θ is the latitude; R_e is the Earth's radius, and R_0 is the deformation radius based on the total depth of the ocean. An asymptotic system of equations is derived and used to study the stability of zonal currents. Three sufficient conditions of stability are obtained, which restrict the slope of the interface between the stratified and non-stratified layers. The results obtained are applied to the subtropical and subarctic frontal currents in the Northern Pacific: the former was found to be stable, the latter was found to be unstable. However, the growth rate of the instability is very small (the effective time of growth is about 2 years).

1. Introduction

The equations which govern stratified flows with large displacement of isopycnal surfaces on the β-plane are very complex and, generally speaking, do not allow analytical study. Even the smallness of the Rossby number (the assumption of geostrophy) does not make them much simpler, as the large number of other parameters does not allow an asymptotic system applicable in all cases to be derived. Thus, one has to consider various regimes in the parameter space of the governing equations and derive separate sets of asymptotic equations for all regimes. For the (simplest) case of two-layer stratification, Cushman-Roisin, Sutyrin & Tang (1992); Benilov (1992) and Benilov & Cushman-Roisin (1994) demonstrated that all of the parameter space can be 'covered' by four relatively simple asymptotic systems. Using these systems, the stability of two-layered shear flows was examined by Benilov (1992, 1995*a*), Swaters (1993) and Benilov & Cushman-Roisin (1994). Two out of the four systems (and the corresponding stability analyses) were generalized for continuous stratification (Benilov 1993, 1994), while one of the remaining two was shown to have no physical applications (see Benilov & Cushman-Roisin 1994). The last (fourth) regime is dealt with in the present paper.

The dynamics of flows with large displacement of isopycnal surfaces (large-amplitude flows) on the β-plane are governed by three non-dimensional parameters:

(i) the Rossby number

$$\epsilon = U/(fL), \tag{1.1}$$

where U is the effective velocity scale, L is the horizontal spatial scale of the motion and f is the Coriolis parameter;

		Weak β-effect: $\alpha \sim \epsilon^{3/2}$	Strong β-effect: $\alpha \sim \epsilon$
(*a*)	$\delta \sim 1$	Benilov 1992	Benilov 1992
	$\delta \sim \epsilon$	As above	Cushman-Roisin *et al.* 1992
	$\delta \sim \epsilon^2$	Cushman-Roisin *et al.* 1992	As above
(*b*)	$\delta \sim 1$	Benilov 1992, instability	Benilov 1992, 1995*a*, stability
	$\delta \sim \epsilon$	As above	Benilov & Cushman-Roisin 1994; Benilov 1995*a*, stability/instability
	$\delta \sim \epsilon^2$	Swaters 1993, stability/instability	As above
(*c*)	$\delta \sim 1$	Benilov 1993, instability	Benilov 1994, stability
	$\delta \sim \epsilon$	As above	?
	$\delta \sim \epsilon^2$	?	As above

TABLE 1. Classification of large-amplitude geostrophic flows: (*a*) two-layer asymptotic equations: (*b*) stability of two-layer flows; (*c*) a combination of (*a*) and (*b*) for continuously stratified flows

(ii) the β-effect number

$$\alpha = (R_0/R_e)\cot\theta, \tag{1.2}$$

where

$$R_0 = (g'H_0)^{1/2}/f, \tag{1.3}$$

$g' = g\,\delta\rho/\rho_0$ is the reduced acceleration due to gravity, ρ_0 and $(\rho_0+\delta\rho)$ are the minimum and maximum values of the density, H_0 is the total depth of the fluid, θ is the latitude, R_e is the Earth's radius (α can be interpreted as the non-dimensional version of the usual β-parameter: $\alpha = R_0\beta/f$);

(iii) the relative depth of the 'active' layer

$$\delta = H_a/H_0, \tag{1.4}$$

where H_a is the depth of the upper layer where the flow and stratification are localized.

The classification of two-layer large-amplitude geostrophic ($\epsilon \ll 1$) flows can be presented on the plane of the parameters $(\alpha/\epsilon, \delta)$ (see table 1*a*), where each cell corresponds to a system of asymptotic equations. The stability properties of large-amplitude flows were found to be closely linked to the above classification (see table 1*b*).

The corresponding results for continuously stratified flows can be presented in the form of the single table 1(*c*). It can be easily demonstrated that the regime with $\delta \sim \epsilon^2$ has no oceanographic applications: given $\epsilon \lesssim 0.1$, it corresponds to $\delta \lesssim 0.01$, while in the real ocean $\delta \sim \frac{1}{2}$–$\frac{1}{15}$.

The present paper examines the stability of large-amplitude geostrophic flows with a strong β-effect and thin upper layer: $\alpha \sim \delta \sim \epsilon$. It is convenient to consider this case within the framework of the 'semi-Lagrangian' variables, where the vertical variable becomes Lagrangian, while the horizontal variables remain Eulerian (§2). The asymptotic governing equations are derived (§3) and demonstrated (§4) to be separable, which corresponds physically to the 'equivalent barotropic mode' observed by Killworth (1992) in his analysis of the FRAM data. The equations derived are used in the stability analysis of zonal flows in §5. The results obtained are applied to the subarctic and subtropical frontal currents in the Northern Pacific (§6). It turns out, however, that the equations derived do not describe the stability of short disturbances, which have to be studied using a different approach (§7).

2. Basic equations

Consider a flow localized in a thin layer of continuously stratified fluid, which overrides a thick homogeneous layer. The form of the interface is unknown, which drastically complicates the problem. Accordingly, we shall use the semi-Lagrangian variables (Odulo 1979), which map the upper layer into a domain with a fixed plane boundary. Moreover, these variables map all isopycnal surfaces into horizontal planes.

In §2.1 we shall introduce the semi-Lagrangian variables for unspecified stratification, and in §2.2 we shall adapt the results obtained for the case of homogeneous lower layer.

2.1. *Semi-Lagrangian variables*

The (non-dimensional) equations which govern stratified flows on the β-plane are

$$\left.\begin{aligned} u_t+uu_x+vu_y+wu_z+p_x &= (1+\alpha y)\,v, \\ v_t+uv_x+vv_y+wv_z+p_y &= -(1+\alpha y)\,u, \end{aligned}\right\} \tag{2.1a}$$

$$p_z = -\rho, \tag{2.1b}$$

$$u_x+v_y+w_z = 0, \tag{2.1c}$$

$$\rho_t+u\rho_x+v\rho_y+w\rho_z = 0. \tag{2.1d}$$

Here
$$t=\tilde{t}f,\quad x=\frac{\tilde{x}}{R_0},\quad y=\frac{\tilde{y}}{R_0},\quad z=\frac{\tilde{z}}{H_0},$$

$$p=\frac{\tilde{p}-g\tilde{z}}{g'H_0},\quad u=\frac{\tilde{u}}{R_0 f},\quad v=\frac{\tilde{v}}{R_0 f},\quad w=\frac{\tilde{w}}{H_0 f},\quad \rho=\frac{\tilde{\rho}-\rho_0}{\delta\rho},$$

where the dimensional variables (the spatial coordinates $(\tilde{x},\tilde{y},\tilde{z})$, the time $\tilde{t}$, the velocity $(\tilde{u},\tilde{v},\tilde{w})$, the pressure $\tilde{p}$ and the density $\tilde{\rho}$) are marked with tildas.

Equations (2.1) are supplemented by the rigid-lid boundary condition

$$w=0 \quad \text{at} \quad z=-1,0. \tag{2.2}$$

We shall assume, for simplicity, that the flow is not bounded horizontally.

Next we introduce the following change of variables $(t,x,y,z)\rightarrow(t',x',y',\xi)$:

$$t=t',\quad x=x',\quad y=y',\quad z=Z(t',x',y',\xi), \tag{2.3}$$

where $\xi\in(-1,0)$ and Z is governed by

$$Z_{t'}+uZ_{x'}+vZ_{y'}-w=0, \tag{2.4}$$

$$\left.\begin{aligned} Z&=-1 \quad \text{at} \quad \xi=-1, \\ Z&=0 \quad \text{at} \quad \xi=0 \end{aligned}\right\} \tag{2.5}$$

((2.4)–(2.5) should be treated as a formal definition of the Lagrangian variable ξ). The boundaries of the fluid correspond to $\xi=-1,0$; it should also be observed that (2.4)–(2.5) automatically satisfy boundary conditions (2.2).

Substituting (2.3) into (2.1) and taking into account (2.4), we obtain

$$\left.\begin{aligned} u_{t'}+uu_{x'}+vu_{y'}+(p_{x'}-(Z_{x'}/Z_\xi)\,p_\xi) &= (1+\alpha y')\,v, \\ v_{t'}+uv_{x'}+vv_{y'}+(p_{y'}-(Z_{y'}/Z_\xi)\,p_\xi) &= -(1+\alpha y')\,u, \end{aligned}\right\} \tag{2.6a}$$

$$(1/Z_\xi)\,p_\xi = -\rho, \tag{2.6b}$$

$$u_{x'}+v_{y'}-(1/Z_\xi)(Z_{x'}u_\xi+Z_{y'}v_\xi-w_\xi) = 0, \tag{2.6c}$$

$$\rho_{t'}+u\rho_{x'}+v\rho_{y'} = 0. \tag{2.6d}$$

Equation (2.6d) can be satisfied by

$$\rho = \rho(\xi), \tag{2.7}$$

which means that the density is 'frozen' into fluid particles and can be identified with the vertical Lagrangian variable ξ. It should also be noted that (2.7) together with (2.5) do not permit outcropping (in the strictest sense of the word). However, large displacements are allowed everywhere in the interior including particles located at an infinitesimal distance from the surface, which is just as good as outcropping.

Next we differentiate (2.4) with respect to ξ, then multiply (2.6c) by Z_ξ and add them. We obtain (primes omitted):

$$n_t + (un)_x + (vn)_y = 0, \tag{2.8}$$

where

$$n = Z_\xi \tag{2.9}$$

characterizes vertical stretching of Taylor's columns. Intending to eliminate Z, we integrate (2.9) with respect to ξ and take into account (2.5):

$$Z = -\int_\xi^0 n(t,x,y,\xi')\,\mathrm{d}\xi', \tag{2.10}$$

$$\int_{-1}^0 n(t,x,y,\xi)\,\mathrm{d}\xi = 1. \tag{2.11}$$

Solving (2.6b), we have

$$p(t,x,y,\xi) = \eta(t,x,y) + \int_\xi^0 \rho(\xi')\,n(t,x,y,\xi')\,\mathrm{d}\xi', \tag{2.12}$$

where η is the pressure on the rigid lid. Now we substitute (2.10) and (2.12) into (2.6a) and obtain (primes omitted):

$$\left.\begin{aligned} u_t + uu_x + vu_y + q_x &= (1+\alpha y)\,v, \\ v_t + uv_x + vv_y + q_y &= -(1+\alpha y)\,u, \end{aligned}\right\} \tag{2.13}$$

where

$$q(t,x,y,\xi) = \eta(t,x,y) + \int_\xi^0 [\rho(\xi') - \rho(\xi)]\,n(t,x,y,\xi')\,\mathrm{d}\xi'. \tag{2.14}$$

Equations (2.8), (2.11) and (2.13)–(2.14) form a closed system for the unknowns u, v, q, η and n. We shall need the relationship of q to the physical pressure p. Comparison of (2.12) to (2.14) yields

$$q = p - \rho Z. \tag{2.15}$$

Equality (2.15) demonstrates that q is a Bernoulli function, i.e. it represents the non-hydrostatic component of the pressure.

2.2. *Flows with a homogeneous lower layer*

We assume that the fluid can be divided into two layers: the upper layer is continuously stratified, the lower layer is homogeneous:

$$\rho(\xi) = \begin{cases} \rho(\xi) & \text{for} \quad \xi > -d, \\ 1 & \text{for} \quad \xi \leqslant -d, \end{cases} \tag{2.16a}$$

where d is the unperturbed depth of the upper layer. It should also be assumed that the flow in the lower layer is vertically homogeneous:

$$[u,v,q,n](t,x,y,\xi) = \begin{cases} [u,v,q,n](t,x,y,\xi) & \text{for} \quad \xi > -d, \\ [U,V,Q,N](t,x,y) & \text{for} \quad \xi \leqslant -d, \end{cases} \tag{2.16b}$$

where the lower-case/capital letters correspond to the upper/lower layer, respectively. Substitution of (2.16) into (2.11) yields

$$\int_{-d}^{0} n(t,x,y,\xi)\,\mathrm{d}\xi+(1-d)\,N(t,x,y)=1. \tag{2.17}$$

Then we substitute (2.16) into (2.8), (2.13)–(2.14) and, putting $\xi \leqslant -d$, obtain

$$N_t+(UN)_x+(VN)_y=0, \tag{2.18a}$$

$$\left.\begin{aligned} U_t+UU_x+VU_y+Q_x &= (1+\alpha y)\,V, \\ V_t+UV_x+VV_y+Q_y &= -(1+\alpha y)\,U, \end{aligned}\right\} \tag{2.18b}$$

$$Q(t,x,y)=\eta(t,x,y)+\int_{-d}^{0}[\rho(\xi)-1]\,n(t,x,y,\xi)\,\mathrm{d}\xi.$$

Finally, we shall eliminate η by differentiating (2.14) with respect to ξ:

$$q_\xi(t,x,y,\xi)=-\rho_\xi\int_{\xi}^{0} n(t,x,y,\xi')\,\mathrm{d}\xi'. \tag{2.19}$$

Then we divide (2.19) by ρ_ξ and differentiate it again:

$$\left(\frac{1}{\rho_\xi}q_\xi\right)_\xi=n. \tag{2.20}$$

This equation should be supplemented by the boundary condition

$$q_\xi=0 \quad \text{at} \quad \xi=0, \tag{2.21}$$

which follows from (2.19), and another one,

$$q=0 \quad \text{at} \quad \xi=-d, \tag{2.22}$$

which follows from the continuity of the pressure across the interface. In order to rewrite the governing equations in a purely differential form, we substitute (2.20) into (2.17) and take into account (2.21):

$$-\frac{1}{\rho_\xi}q_\xi+(1-d)\,N(t,x,y)=1 \quad \text{at} \quad \xi=-d. \tag{2.23}$$

The complete system of governing equations consists of upper-layer equations (2.8), (2.13), (2.20)–(2.21); coupling equations (2.22)–(2.23); lower-layer equations (2.18).

3. Asymptotic analysis of the governing equations

It is convenient to introduce the depth of the upper layer

$$h=\int_{-d}^{0} n(t,x,y,\xi)\,\mathrm{d}\xi \tag{3.1a}$$

(see (2.9) and (2.5)) and eliminate the constant d by expanding the range of ξ from $[-d,0]$ to $[-1,0]$:

$$\hat{\xi}=\xi/d, \quad \hat{n}=nd. \tag{3.1b}$$

Substituting (3.1) into the governing equations (2.8), (2.13), (2.20)–(2.23), (2.18) and omitting hats; we obtain

$$\left.\begin{aligned} u_t+uu_x+vu_y+q_x &= (1+\alpha y)v \\ v_t+uv_x+vv_y+q_y &= -(1+\alpha y)u \\ \left(\frac{1}{\rho_\xi}q_\xi\right)_\xi &= n \\ n_t+(un)_x+(vn)_y &= 0 \end{aligned}\right\} \text{ for } \xi\in(-1,0); \tag{3.2a}$$

$$q_\xi = 0 \quad \text{at} \quad \xi = 0; \tag{3.2b}$$

$$q = Q \quad \text{at} \quad \xi = -1; \tag{3.2c}$$

$$\frac{1}{\rho_\xi}q_\xi = -h \quad \text{at} \quad \xi = -1; \tag{3.2d}$$

$$\left.\begin{aligned} U_t+UU_x+VU_y+Q_x &= (1+\alpha y)V, \\ V_t+UV_x+VV_y+Q_y &= -(1+\alpha y)U, \\ -h_t+[U(1-h)]_x+[V(1-h)]_y &= 0. \end{aligned}\right\} \tag{3.2e}$$

It should be recalled here that (u,v,q) and (U,V,Q) are the horizontal velocity and Bernoulli function in the upper and lower layers, respectively, ξ is the vertical Lagrangian variable ($\xi=-1,0$ correspond to the boundaries of the upper layer), n is the derivative of the vertical displacement of fluid particles with respect to ξ and characterizes the stretching of Taylor's columns, and h is the depth of the (continuously stratified) upper layer. It is also worth noting that, in terms of the semi-Lagrangian variables, the density ρ depends only on ξ (fluid particles cannot change their densities).

Scaling of equations (3.2) is very similar to the two-layer case (see Benilov & Cushman-Roisin 1994). The horizontal spatial scale is comparable to R_0, therefore x and y should not be scaled at all:

$$x=\hat{x}, \quad y=\hat{y}. \tag{3.3a}$$

The displacement of isopycnal surfaces is large:

$$\xi=\hat{\xi}, \quad n=\hat{n}. \tag{3.3b}$$

We shall consider geostrophic flows, where the Rossby number ϵ is small:

$$u=\epsilon\hat{u}, \quad v=\epsilon\hat{v}, \quad q=\epsilon\hat{q}. \tag{3.3c}$$

The upper layer is assumed thin:

$$h=\epsilon\hat{h} \tag{3.3d}$$

and the flow in the lower layer is weak:

$$U=\epsilon^2\hat{U}, \quad V=\epsilon^2\hat{V}, \quad Q=\epsilon^2\hat{Q}. \tag{3.3e}$$

The β-effect is assumed strong:

$$\alpha=\epsilon\hat{\alpha}. \tag{3.3f}$$

Finally, we should estimate the effective time scale of the flow: using the dispersion relations of the linear Rossby waves and taking into account (3.3a,d,f), we have

$$\omega_{bt}=\frac{\alpha k_{(x)}}{k^2}\sim\epsilon, \quad \omega_{bc}=\frac{\alpha k_{(x)}}{k^2+h^{-1}}\sim\epsilon^2;$$

where ω_{bt} and ω_{bc} are the (non-dimensional) barotropic and baroclinic frequencies, respectively. We are interested in the baroclinic motion and, accordingly, put

$$t = \epsilon^{-2}\hat{t}. \tag{3.3g}$$

Substitution of (3.3) into (3.2) yields (hats omitted)

$$\left.\begin{aligned} \epsilon^2 u_t + \epsilon(uu_x + vu_y) + q_x &= (1+\epsilon\alpha y)\, v \\ \epsilon^2 v_t + \epsilon(uv_x + vv_y) + q_y &= -(1+\epsilon\alpha y)\, u \end{aligned}\right\} \text{ for } \xi\in(-1,0); \tag{3.4a}$$

$$\epsilon n_t + (un)_x + (vn)_y = 0 \quad \text{for} \quad \xi\in(-1,0); \tag{3.4b}$$

$$n = \left(\frac{1}{\rho_\xi} q_\xi\right)_\xi \quad \text{for} \quad \xi\in(-1,0), \tag{3.4c}$$

$$q_\xi = 0 \quad \text{at} \quad \xi = 0; \tag{3.4d}$$

$$q = \epsilon Q \quad \text{at} \quad \xi = -1; \tag{3.4e}$$

$$\frac{1}{\rho_\xi} q_\xi = -h \quad \text{at} \quad \xi = -1; \tag{3.4f}$$

$$\left.\begin{aligned} \epsilon^2(U_t + UU_x + VU_y) + Q_x &= (1+\epsilon\alpha y)\, V, \\ \epsilon^2(V_t + UV_x + VV_y) + Q_y &= -(1+\epsilon\alpha y)\, U, \end{aligned}\right\} \tag{3.4g}$$

$$-\epsilon h_t + [U(1-\epsilon h)]_x + [V(1-\epsilon h)]_y = 0. \tag{3.4h}$$

First we expand (3.4*a*) and (3.4*g*) into power series in ϵ:

$$\left.\begin{aligned} v &= q_x - \epsilon[J(q, q_y) + \alpha y q_x] + O(\epsilon^2), \\ u &= -q_y - \epsilon[J(q, q_x) - \alpha y q_y] + O(\epsilon^2); \end{aligned}\right\} \tag{3.5a}$$

$$\left.\begin{aligned} V &= Q_x - \epsilon\alpha y Q_x + O(\epsilon^2), \\ U &= -Q_y + \epsilon\alpha y Q_y + O(\epsilon^2); \end{aligned}\right\} \tag{3.5b}$$

where $J(p,q) = p_x q_y - p_y q_x$ is the Jacobian operator. Then we substitute (3.5*a*) and (3.4*c*) into (3.4*b*):

$$\epsilon\left(\frac{1}{\rho_\xi} q_\xi\right)_{\xi t} + (1+\epsilon\alpha y)\, J\left[q, \left(\frac{1}{\rho_\xi} q_\xi\right)_\xi\right] - \epsilon\alpha q_x \left(\frac{1}{\rho_\xi} q_\xi\right)_\xi - \epsilon\nabla\cdot\left[\left(\frac{1}{\rho_\xi} q_\xi\right)_\xi J(q, \nabla q)\right] = O(\epsilon^2). \tag{3.6}$$

Substitution of (3.5*b*) into (3.4*h*) yields

$$\epsilon h_t + \epsilon J(Q, h) + \epsilon\alpha Q_x = O(\epsilon^2). \tag{3.7}$$

Equations (3.6)–(3.7) and (3.4*d*–*f*) form a closed system for q, h and Q. In order to derive the zero-order equations, we integrate (3.6) with respect to ξ over $(-1,0)$. Integrating by parts and taking into account (3.4*d*–*f*), we get

$$\epsilon h_t + \epsilon J(Q,h) + \epsilon\alpha \int_{-1}^{0} q_{\xi x}\left(\frac{1}{\rho_\xi} q_\xi\right) \mathrm{d}\xi - \epsilon\nabla\cdot\int_{-1}^{0}\left(\frac{1}{\rho_\xi} q_\xi\right)_\xi J(q, \nabla q)\, \mathrm{d}\xi = O(\epsilon^2). \tag{3.8}$$

Omitting small terms in (3.7), we obtain

$$h_t + J(Q,h) + \alpha Q_x = 0. \tag{3.9a}$$

Omitting small terms in (3.8), dividing it by ϵ and subtracting from it (3.9*a*), we have

$$\alpha\left[Q - \frac{1}{2}\int_{-1}^{0}\frac{1}{\rho_\xi}(q_\xi)^2\, \mathrm{d}\xi\right]_x + \nabla\cdot\int_{-1}^{0}\left(\frac{1}{\rho_\xi} q_\xi\right)_\xi J(q, \nabla q)\, \mathrm{d}\xi = 0. \tag{3.9b}$$

Finally, we omit small terms in (3.6):

$$J\left[q, \left(\frac{1}{\rho_\xi} q_\xi\right)_\xi\right] = 0. \tag{3.9c}$$

Equation (3.9*c*) should be supplemented by the (zero-order version of) boundary conditions (3.4*d–f*):

$$q_\xi = 0 \quad \text{at} \quad \xi = 0, \tag{3.9d}$$

$$q = 0 \quad \text{at} \quad \xi = -1, \tag{3.9e}$$

$$\frac{1}{\rho_\xi} q_\xi = -h \quad \text{at} \quad \xi = -1. \tag{3.9f}$$

Equations (3.9) form a closed system for $q(t, x, y, \xi)$, $h(t, x, y)$ and $Q(t, x, y)$. It is worth noting that the only evolutionary variable is h, whereas Q and q 'adjust' themselves to h through the non-evolutionary equations (3.9*b*) and (3.9*c–f*). This implies that the density field evolves quasi-statistically.

To conclude this subsection, we shall rewrite equations (3.9) in terms of the original (Eulerian) variables. First we shall recall that the velocities and pressure in the active (upper) layer are given by

$$u = -q_y, \quad v = q_x. \tag{3.10}$$

$$p = q - \rho Z \tag{3.11}$$

(see (3.5*a*) and (2.15)). We shall also need the relationship between q and Z:

$$Z = \frac{1}{\rho_\xi} q_\xi \tag{3.12}$$

(see (2.20) and (2.9)). The relationships between the Eulerian and semi-Lagrangian derivatives are

$$q_\xi = \frac{1}{(\xi_z)_{(E)}} (q_z)_{(E)}, \quad q_x = (q_x)_{(E)} - \frac{(\xi_x)_{(E)}}{(\xi_z)_{(E)}} (q_z)_{(E)}, \quad q_x = (q_x)_{(E)} - \frac{(\xi_x)_{(E)}}{(\xi_z)_{(E)}} (q_z)_{(E)}, \tag{3.13}$$

where the subscript (E) marks Eulerian derivatives. Substituting (3.11) and (3.13) into (3.10) and taking into account that

$$(p_z)_{(E)} = -\rho \tag{3.14}$$

(see (2.1*b*)) and

$$\xi = \xi(\rho) \tag{3.15}$$

(see (2.7)), we have

$$u = -(p_y)_{(E)}, \quad v = (p_x)_{(E)}. \tag{3.16}$$

Using (3.10) and (3.12), we rewrite equation (3.9*c*) as

$$u Z_{\xi x} + v Z_{\xi y} = 0. \tag{3.17}$$

Substitution of (3.13) and (3.16) into (3.17) yields

$$\left[\left(\frac{1}{\xi_z} J(p, \xi)\right)_z\right]_{(E)} = 0. \tag{3.18}$$

Substituting (3.15) into (3.18), we replace ξ by ρ and integrate (3.18):

$$\frac{1}{\rho_\zeta} J(p, \rho) = \text{const.}$$

Determining the constant from the condition

$$\rho = 0 \quad \text{at} \quad \zeta = 0,$$

and making use of (3.14), we obtain (subscript (E) dropped):

$$J(p, p_z) = 0. \tag{3.19a}$$

In terms of the Eulerian variables the boundary conditions (3.9*d*–*f*) become

$$p_z = 0 \quad \text{at} \quad z = 0, \tag{3.19b}$$

$$p_z = 1 \quad \text{at} \quad z = -h, \tag{3.19c}$$

$$p = -h \quad \text{at} \quad z = -h \tag{3.19d}$$

(observe that (3.19*b*, *c*) follow from (3.14) and the conditions $\rho = 0, 1$ at $z = 0, -1$). We shall rewrite equation (3.9*a*) in terms of the pressure P in the lower layer:

$$h_t + J(P, h) + \alpha(P-h)_x = 0, \tag{3.19e}$$

where $P = Q + h$. Finally, integrating the first integral in (3.9*b*) by parts and making use of (3.11)–(3.15) and (3.19*a*), we obtain

$$\alpha\left(Q - \tfrac{1}{2}h^2 + \int_{-h}^{0} p\,\mathrm{d}z\right)_x + \nabla\cdot\int_{-h}^{0} J(p, \nabla p)\,\mathrm{d}z = 0. \tag{3.19f}$$

Compared to the semi-Lagrangian equations (3.9), the main disadvantage of the Eulerian system (3.19) is the variable lower limit in the integrals in (3.19*f*).

4. Equivalent barotropic mode

Analysing the mean-flow field computed by the Fine Resolution Antarctic Model (FRAM), Killworth (1992) observed that, 'to a good degree of approximation, much of the horizontal velocity field behaves as if there were an equivalent barotropic flow. In other words, the flow at one depth is both parallel and proportional to the flow at another depth, despite the complications of realistic topography, eddies, and so on.' The direct measurements available (Sciremammano 1979; Bryden & Heath 1985) also give evidence for the existence of a well-correlated vertical structure in large-scale oceanic currents.

A mathematical explanation of the existence of the equivalent-barotropic mode (EBM) was put forward by Benilov (1994) for the regime of strong β-effect and thick upper layer. In the present paper, we shall address this question for the regime of strong β-effect and thin upper layer.

From a mathematical viewpoint, EBM corresponds to a separable solution of equations (3.9):

$$q(t, x, \xi) = h(t, x, y)\,\phi(\xi), \tag{4.1a}$$

where $\phi(\xi)$ satisfies the following boundary conditions:

$$\phi_\xi = 0 \quad \text{at} \quad \xi = 0, \tag{4.1b}$$

$$\phi = 0 \quad \text{at} \quad \xi = -1, \tag{4.1c}$$

$$\frac{1}{\rho_\xi}\phi_\xi = -1 \quad \text{at} \quad \xi = -1. \tag{4.1d}$$

Substitution of (4.1) into (3.9) yields

$$\left.\begin{aligned} h_t + J(Q, h) + \alpha Q_x &= 0, \\ \alpha(Q - \nu h^2)_x + \gamma\nabla\cdot[hJ(h, \nabla h)] &= 0, \end{aligned}\right\} \tag{4.2}$$

where $$\nu = \frac{1}{2}\int_{-1}^{0} \frac{1}{\rho_\xi}(\phi_\xi)^2\,d\xi, \quad \gamma = \int_{-1}^{0} \left(\frac{1}{\rho_\xi}\phi_\xi\right)_\xi \phi^2\,d\xi.$$

Although (4.2) is a two-dimensional system, it describes three-dimensional motion of the fluid.

Remarkably, the Eulerian form of the governing equations (3.19) does not admit separable solutions! Mathematically, this occurs because the Eulerian analogue of (4.1*a*),

$$q(t,x,y,\xi) = h(t,x,y)\,\phi(z),$$

is not consistent with the boundary condition (3.19*d*), unless $\phi = 1$ for all z inside the range of $h(t,x,y)$:

$$\phi(z) = 1 \quad \text{for} \quad z \in [-h_{max}, -h_{min}].$$

At the same time, the regime of strong β-effect and thick upper layer (Benilov 1994) supports only the Eulerian version of EBM. Thus, it might be interesting to estimate the correlation of the FRAM velocity field at different densities rather than at different depths (as has been done by Killworth 1992). In some cases, the former can be correlated better than the latter. It should also be interesting to estimate vertical correlation of an instant field (Killworth 1992 worked with the mean flow averaged over long period of time).

It is also worth noting that the conclusion of Sciremammano (1979), Dryden & Heath (1985) and Killworth (1992) contradicts that of Vasilenko & Mirabel (1977) and Inoue (1985), who observed two modes (one barotropic and one baroclinic) in large-scale oceanic currents. However, the latter conclusion can also be accounted for by the two-mode solution found by Benilov (1993) for the regime of weak β-effect and thick upper layer. Generally speaking, one- or two-mode solutions could correspond to different areas of the ocean (different regimes). It should be emphasized, however, that no solution has been found which would describe self-contained nonlinear interaction of three, four or any further finite number of modes.

To conclude this section we adapt equations (3.2) for the two-layer stratification, which corresponds to

$$\rho \to \begin{cases} 0 & \text{for} \quad \xi \in (-1, 0), \\ 1 & \text{for} \quad \xi = -1. \end{cases} \tag{4.3}$$

Assuming that $$\phi(\xi) = 1 + \xi\rho(\xi) - \int_{-1}^{\xi} \rho(\xi')\,d\xi',$$

which satisfies (4.1*b–d*) and, at the same time, corresponds, in the limit $\rho \to 0$, to a vertically homogeneous current in the upper layer:

$$\lim_{\rho\to 0} \phi = 1 \quad \text{for all} \quad \xi \neq -1;$$

we have $$\lim_{\rho\to 0} \nu = -\tfrac{1}{2}, \quad \lim_{\rho\to 0} \gamma = 1. \tag{4.4}$$

System (4.2), (4.4) coincides with the two-layer equations in Benilov & Cushman-Roisin (1994). It should be emphasized that, apart from (insignificant) difference in the values of ν and γ, the two-layer case is very similar to the general case. Given that the equivalent barotropic equations for other large-amplitude geostrophic regimes can also be reduced to the corresponding two-layer equations (Benilov 1993, 1994), we conclude that the two-layer model provides a qualitatively correct approximation for all large-amplitude flows.

5. Stability of zonal flows

System (3.9) admits the following solution:

$$q = \bar{q}(y,\xi), \quad Q = \bar{Q}(y), \quad h = \bar{h}(y); \tag{5.1a}$$

$$\bar{q}_\xi = 0 \quad \text{at} \quad \xi = 0, \tag{5.1b}$$

$$\bar{q} = 0 \quad \text{at} \quad \xi = -1, \tag{5.1c}$$

$$\frac{1}{\rho_\xi}\bar{q}_\xi = -\bar{h} \quad \text{at} \quad \xi = -1, \tag{5.1d}$$

which describes a steady zonal flow with both vertical and horizontal shear. We shall assume that the flow is localized in the y-direction:

$$\bar{q}_y, \bar{h}_y, \bar{Q}_y \to 0 \quad \text{as} \quad y \to \pm\infty.$$

In §5.1, a set of sufficient stability conditions for solution (5.1) will be derived, and in §5.2 we shall discuss what happens if these conditions are violated.

5.1. *Sufficient conditions for stability of zonal flows*

Linearizing (3.9) against the background of (5.1), we seek a harmonic-wave solution:

$$q(t,x,y,\zeta) = \bar{q}(y,\xi) + q(y,\xi)\exp[ik(ct-x)],$$
$$Q(t,x,y) = \bar{Q}(y) + Q(y)\exp[ik(ct-x)],$$
$$h(t,x,y) = \bar{h}(y) + h(y)\exp[ik(ct-x)],$$

where k and c are respectively the wavenumber and the phase speed of disturbances. Substituting these equalities into (3.9) and omitting nonlinear terms, we obtain

$$(c+\bar{Q}_y)h - (\alpha+\bar{h}_y)Q = 0, \tag{5.2a}$$

$$\alpha\left[Q - \int_{-1}^{0}\frac{1}{\rho_\xi}\bar{q}_\xi q_\xi \,\mathrm{d}\xi\right]_x + \frac{\partial}{\partial y}\int_{-1}^{0}\left(\frac{1}{\rho_\xi}\bar{q}_\xi\right)_\xi (q\bar{q}_{yy} - q_y\bar{q}_y)\,\mathrm{d}\xi + k^2\int_{-1}^{0}\left(\frac{1}{\rho_\xi}\bar{q}_\xi\right)_\xi \bar{q}_y q\,\mathrm{d}\xi = 0, \tag{5.2b}$$

$$\left.\begin{aligned} q\left(\frac{1}{\rho_\xi}\bar{q}_\xi\right)_{\xi y} - q_y\left(\frac{1}{\rho_\xi}\bar{q}_\xi\right)_\xi &= 0,\\ q_\xi = 0 \quad &\text{at} \quad \xi = 0,\\ q = 0 \quad &\text{at} \quad \xi = -1,\\ \frac{1}{\rho_\xi}q_\xi = -h \quad &\text{at} \quad \xi = -1.\end{aligned}\right\} \tag{5.2c}$$

Equations (5.2*a*, *c*) can be readily solved:

$$h = A(y)\bar{h}_y, \quad Q = \frac{c+\bar{Q}_y}{\alpha+\bar{h}_y}A(y)\bar{h}_y, \quad q = A(y)\bar{q}_y, \tag{5.3}$$

where $A(y)$ is an undetermined function which describes the horizontal structure of the perturbation. Substitution of (5.3) into (5.2*b*) yields an equation for $A(y)$:

$$-(FA_y)_y + (k^2F+G)A + \alpha\bar{h}_y\frac{c+\bar{Q}_y}{\alpha+\bar{h}_y}A = 0, \tag{5.4}$$

where
$$F(y) = \int_{-1}^{0}\left(\frac{1}{\rho_\xi}\bar{q}_\xi\right)_\xi(\bar{q}_y)^2\,\mathrm{d}\xi, \quad G(y) = -\tfrac{1}{2}\alpha\frac{\mathrm{d}}{\mathrm{d}y}\int_{-1}^{0}\frac{1}{\rho_\xi}(\bar{q}_\xi)^2\,\mathrm{d}\xi. \tag{5.5a, b}$$

We assume that the disturbance is localized zonally:

$$A \to 0 \quad \text{as} \quad y \to \pm\infty. \tag{5.6}$$

Now we integrate (5.4) with respect to y over $(-\infty, \infty)$ and, integrating by parts, take into account (5.6):

$$l_1 + l_2 c = 0,$$

where
$$l_1 = \int_{-1}^{0} \left[F|A_y|^2 + \left(k^2 F + G + \frac{\bar{h}_y \bar{Q}_y}{\alpha + \bar{h}_y} \right) |A|^2 \right] dy,$$

$$l_2 = \alpha \int_{-1}^{0} \frac{\bar{h}_y}{\alpha + \bar{h}_y} |A|^2 \, dy.$$

Since $\operatorname{Im} l_1 = \operatorname{Im} l_2 = 0$, the condition

$$l_2 \neq 0, \infty \tag{5.7}$$

guarantees that $c = -l_1/l_2$ is real. Finally, the validity of (5.7) can be ensured by

$$\bar{h}_y(y) \geqslant 0 \tag{5.8a}$$

or
$$-\alpha < \bar{h}_y(y) \leqslant 0 \tag{5.8b}$$

or
$$\bar{h}_y(y) < -\alpha. \tag{5.8c}$$

Conditions (5.8) are sufficient stability criteria.

5.2. *Instability of zonal flows*

It is convenient to rewrite (5.8) as follows:

$$\alpha + \bar{h}_y(y) \text{ does not change sign} \tag{5.9a}$$

and
$$\bar{h}_y(y) \text{ does not change sign.} \tag{5.9b}$$

As always, it is very difficult to rigorously prove stability or instability of flows that do *not* satisfy any of the sufficient criteria. However, there is some (analytical and numerical) evidence that violation of condition (5.9*a*) destabilizes the flow, while violation of condition (5.9*b*) does not.

First we consider what happens if condition (5.9*a*) is violated and $\alpha + \bar{h}_y$ changes at, say, $y = y_1$:

$$\bar{h}_y(y_1) = -\alpha.$$

As the denominator of the last term on the left-hand side of (5.4) vanishes, y_1 is a singular point of this equation. In the two-layer case, this singularity can be eliminated (regularized) by taking into account higher-order ageostrophic corrections in the lower layer (Benilov 1995*a*). Using the same approach, one can obtain the regularized version of equation (5.4) for the case of continuous stratification:

$$-(FA'_y)_y + (k^2 F + G) A' + \alpha \bar{h}_y \frac{c + \bar{Q}_y}{\alpha + \bar{h}_y + \mu c} A' = 0, \tag{5.10a}$$

where μ is a small, but finite, positive number (it depends on the parameters of the lower layer and is proportional to the Rossby number; as we take the limit $\mu \to 0$, the specific expression for μ is not important). Clearly, if $\operatorname{Im} c \neq 0$, the singularity disappears, and the regularized solution to the original equation (5.4) is given by

$$A = \lim_{\mu \to 0} A'. \tag{5.10b}$$

Using the standard Wronskian method (e.g. Dikiy 1976; Benilov 1995*a*), one can prove that the regularized eigenvalue problem (5.10), (5.6) may have real (stable) eigenvalues only for isolated values of k. In other words, if (5.10), (5.6) have a solution for any continuous interval $[k_1, k_2]$, the flow is unstable. Although we have no rigorous proof of the existence of a solution to (5.10), (5.6), numerical results (see §6) indicate that there exists at least one unstable mode in all cases.

Consider now a point, say y_2, where $\bar{h}_y$ changes sign:

$$\bar{h}_y(y_2) = 0$$

and condition (5.9*b*) is violated. Generally speaking, y_2 is a regular point of equation (5.4) and therefore does not need regularization. Using a perturbation method similar to that of Griffiths, Killworth & Stern (1982) and Benilov (1995*a*), one can prove the existence of unstable flows in the limit $\alpha \rightarrow 0$. As we consider the case of strong β-effect, this result is of no interest; whereas the numerical simulation (Benilov 1995*a* and §6 of the present paper) suggests that flows with finite α are stable for all flows except two-layered ones. In order to explain this result, we adapt equation (5.4) for the two-layer stratification (4.3). Assuming that

$$\bar{q}(y, \xi) = \bar{h}(y)\left[1 - \xi\rho(\xi) - \int_{-1}^{\xi} \rho(\xi')\,\mathrm{d}\xi'\right], \tag{5.11}$$

which corresponds to a two-layer current, we substitute (4.3) and (5.11) into (5.5) and obtain

$$F(y) = \bar{h}(\bar{h}_y)^2, \tag{5.12}$$

$$G(y) = \alpha\bar{h}\bar{h}_y. \tag{5.13}$$

The reason for instability is clear now: in the two-layer case, the coefficient $F(y)$ of the highest derivative in equation (5.4) is proportional to $\bar{h}_y$ and vanishes at y_2. As a result, y_2 is a singular point, similar to y_1, and therefore destabilizes the flow. Benilov (1995*a*) demonstrated that the singularity at y_2 can be regularized by taking into account the higher ageostrophic corrections or viscosity in the upper layer. We shall not dwell on this in detail, as any deviation from the two-layer stratification makes $F(y)$ strictly positive and eliminates the instability. In order to illustrate this, $F(y)$ will be rewritten in the Eulerian variables:

$$F(y) = \int_{-\bar{h}}^{0} \bar{u}^2(y, z)\,\mathrm{d}z,$$

where $\bar{u}(z, y)$ is the mean velocity profile in the upper layer. Thus, $F(y)$ is the horizontal density of the kinetic energy in the upper layer and does not vanish unless $\bar{u}(z, y_2) = 0$ for all z values. In order to include the two-layer model in the general case, we assume that it has an infinitesimal continuous correction such that

$$F(y) = \bar{h}[(\bar{h}_y)^2 + \text{const}], \quad 0 < \text{const} \ll 1 \tag{5.14}$$

(compare (5.14) to (5.12)). This form of $F(y)$ eliminates the (physically meaningless) singularity at $y = y_2$, and the two-layer flows with sign-indefinite $\bar{h}_y$ are likely to be stable just as continuously stratified flows are.

6. The subarctic and subtropical frontal currents in the Northern Pacific

In this section, the above stability analysis will be applied to the subarctic and subtropical frontal currents in the Northern Pacific. According to Roden's (1976) experimental data, the latter flow consists of two eastward jets (axes located at 27° 30′ N and 31° 30′ N) and a weaker westward jet in between (see figure 9 of Roden's paper). In what follows, we shall use the following notation: SA = subarctic frontal

	SA	ST_1	ST_2	ST_3
ϵ	0.021	0.016	0.021	0.050
α	0.015	0.031	0.034	0.044
δ	0.091	0.064	0.064	0.091
$\bar{h}_y$	−0.024	−0.014	0.019	−0.025

TABLE 2. Parameters of the jets in the Northern Pacific (from Benilov 1995*b*) ϵ, α and δ are defined by (1.1), (1.2) and (1.4); and $\bar{h}_y$ is the characteristic slope of the interface

current; ST_1 = subtropical frontal current, northern (eastward) jet; ST_2 = subtropical frontal current, middle (westward) jet; ST_3 = subtropical frontal current, southern (eastward) jet.

The parameters of the jets estimated by Benilov (1995*b*) are shown in table 2, which demonstrates that:

(i) all three jets can be treated as flows with a thin upper layer and strong β-effect ($\epsilon \sim \alpha \sim \delta$);

(ii) SA is likely to be unstable ($\alpha + \bar{h}_y$ changes sign);

(iii) if considered separately, ST_1, ST_2 and ST_3 are all stable ($\alpha + \bar{h}_y > 0, \bar{h}_y > 0$ and $\alpha + \bar{h}_y > 0$, respectively);

(iv) if considered as a single jet, ($ST_1 + ST_2$) and ($ST_2 + ST_3$) violate criterion (5.9*b*), but are likely to be stable as they do not violate the crucial condition (5.9*a*).

The growth rate of the instability of the subarctic frontal current was calculated using the regularized eigenvalue problem (5.10), (5.6) and the two-layer model of stratification (5.12)–(5.13). The horizontal profile of the flow was modelled by

$$\bar{h} = \delta[1-(y/l)^2], \quad y_1 < y < y_2, \tag{6.1a}$$

where $\delta = 0.091$, $l = 2.97$, $y_1 = 0$, and $y_2 = 0.77$ ((6.1*a*) corresponds dimensionally to

$$\tilde{\bar{h}} = (500\text{ m})\left[1-\left(\frac{\tilde{y}}{260\text{ km}}\right)^2\right], \quad 0 < \tilde{y} < 200\text{ km}$$

see table 1 in (Benilov 1995*b*)). We assume that there is no flow in the lower layer:

$$\bar{Q} = 0. \tag{6.1b}$$

The eigenvalue problem (5.10), (5.6), (5.12)–(5.13), (6.1) was integrated numerically. The results demonstrate that the solution exists for a continuous interval of wavenumbers k, where the eigenvalue c is complex and corresponds to instability. The growth rate $[k \operatorname{Im} c(k)]$ of the instability is plotted in figure 1(*a*), which demonstrates that the instability is very weak – the time of strongest growth is approximately 20.8 months. Figure 1(*a*) also demonstrates that the instability takes place in the long-wave region, i.e. the wavelength of unstable perturbations is larger than the width of the flow. Figure 1(*b*), in turn, shows that the unstable perturbations propagate very slowly (slower than 0.6 cm s^{-1}).

It should be noted that the small growth rate and phase speed of the instability came as no surprise, as the dynamics of flows with thin upper layer must have the same time scale as the evolution of oceanic lenses, which have approximately the same parameters and are known to preserve their form for years.

Finally, we examined the stability of $ST_1 + ST_2$ as a single jet within the framework of the stability boundary-value problem (5.4), (5.6), (5.13)–(5.14). Although this current does not satisfy criterion (5.9*b*), it was found to be stable. This should have been expected, as ($ST_1 + ST_2$) does not violate the crucial condition (5.9*a*).

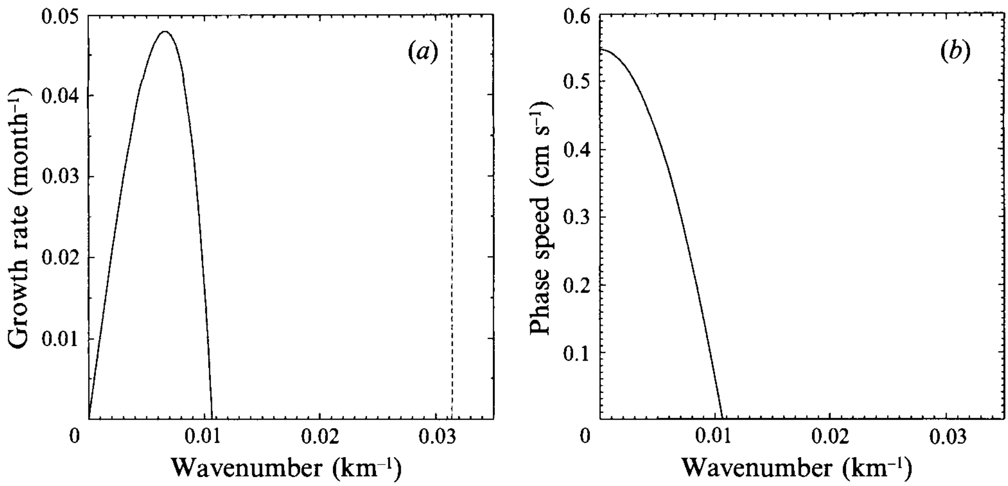

FIGURE 1. Instability of the subarctic frontal current: (*a*) growth rate *vs*. wavenumber (vertical dashed line corresponds to $k = \pi/L$, where $2L = 200$ km is the width of the mean flow); (*b*) phase speed *vs*. wavenumber.

7. Short disturbances

It should be noted that use of scaled equations like (3.9) in a stability analysis is always subject to the criticism that possible instabilities have been scaled out of the problem. Moreover, in our case such doubts are fortified by the fact that asymptotic equations (3.9) describe only long disturbances.

A restriction for the horizontal spatial scale L of the motion follows from the condition of geostrophy:

$$\epsilon = U/(fL) \ll 1, \tag{7.1}$$

where U is the effective velocity scale and f is the Coriolis parameter. Indeed, taking into account the geostrophic relation

$$U = g' \, \delta H_a / L, \tag{7.2a}$$

where we assume that δH_a, the depth variation of the upper layer, is comparable to H_a (large-amplitude flow):

$$\delta H_a \sim H_a. \tag{7.2b}$$

Substitution of (7.2) into (7.1) yields

$$g' H_a / (f^2 L^2) \ll 1.$$

Given that the wavelength of disturbances and the width of the flow are of the same order, this inequality restricts them both:

$$L^2 \gg R_a^2, \tag{7.3}$$

where

$$R_a = (g' H_a)^{1/2} / f \tag{7.4}$$

is the deformation radius based on the depth of the upper layer (compare (7.4) to (1.3)). We conclude that disturbances whose wavelengths do not satisfy (7.3) need separate consideration.

First of all, we observe that, if the wavelength of a disturbance is much smaller than the effective spatial scale of the mean flow, the stability analysis can be carried out locally in the approximation of *small*-amplitude flows. Indeed, variations of the mean-flow parameters over the wavelength of a short perturbation are much smaller than their local values. Accordingly, we can make use of the quasi-geostrophic equations and assume that the mean flow is horizontally homogeneous.

This approximation was considered by Benilov (1995*b*), who demonstrated that (7.3) is valid only if the width L of the mean flow is comparable to the wavelength λ of the disturbance: $L \sim \lambda$. If, however, $\lambda \ll L$ (which is what we want to look at), then the condition, restricting long-wave approximation, is, in fact, stronger than (7.3):

$$\text{long disturbances:} \quad \lambda^2 \gtrsim R_0^2, \tag{7.5a}$$

where R_0 is the deformation radius based on the total depth of the fluid (see (1.3)). Apart from this, two ranges of shorter disturbances were introduced:

$$\text{medium disturbances:} \quad R_0^2 \gg \lambda^2 \gg R_a^2, \tag{7.5b}$$

$$\text{short disturbances:} \quad \lambda^2 \lesssim R_a^2. \tag{7.5c}$$

For medium disturbances, the necessary and sufficient stability criterion has been derived, which turned out to coincide with the necessary condition (5.8*b*) (see condition (4.14) of Benilov (1995*b*), with $s = \bar{h}_y$). For short disturbances, an additional necessary condition of stability has been derived (see condition (5.15) of Benilov (1995*b*), with $u = -\bar{q}_y$):

$$\left.\begin{aligned} &\left(\frac{1}{\rho_\xi}\bar{q}_{y\xi}\right)_\xi \quad \text{does not change sign,} \\ &\bar{q}_{y\xi}(0) \quad \text{has the same sign as} \quad \left(\frac{1}{\rho_\xi}\bar{q}_{y\xi}\right)_\xi . \end{aligned}\right\} \tag{7.6}$$

The long/medium-wave (sufficient and necessary) criterion (5.8*b*) and the short-wave (necessary) condition (7.6) seem to provide a reasonably complete picture of stability for all three ranges of disturbances (7.5*a–c*). It is also worth noting that the *medium*-wave instability of the subarctic flow is much stronger than the *long*-wave instability (compare the results of Benilov 1995*b* to the results of this paper).

It should be emphasized, however, that Benilov's (1995*b*) results were obtained for an idealized case of a horizontally homogeneous ocean and are applicable to the present case only if the frontal flow can be assumed much wider than the wavelengths of medium/short disturbances. Unfortunately, this assumption is of limited relevance to the real ocean. First, the ratio of the 'medium wavelength' to the width of the flow is proportional to the fourth root of the small parameter:

$$\lambda_{medium}/L \sim (H_a/H_0)^{1/4}.$$

As a result, wavelengths of medium disturbances are only marginally smaller than the typical width of oceanic currents (or, given a factor of 2π, even comparable to it!). Secondly, a sufficiently strong horizontal shear can dramatically change the stability properties of a flow (e.g. Barcilon & Blumen 1995). Thus, the results of the present paper on medium/short disturbances should be perceived only as a quantitative estimate.

8. Conclusions

In this paper, we have considered the stability of large-amplitude geostrophic flows localized in a thin continuously stratified layer which overrides a thick homogeneous layer. The β-effect was assumed strong, i.e.

$$\alpha \gtrsim \epsilon,$$

	$\alpha \sim \epsilon^{3/2}$	$\alpha \sim \epsilon$
$\delta \sim 1$	Benilov 1993, two-mode Eulerian solution, or two-mode semi-Lagrangian solution	Benilov 1994, one-mode Eulerian solution
$\delta \sim \epsilon$	As above	Present paper, one-mode semi-Lagrangian solution
$\delta \sim \epsilon^2$	?	As above

TABLE 3. The equivalent-barotropic mode in large-amplitude geostrophic flows

where the Rossby number ϵ and the β-effect number α are defined by (1.1) and (1.2), respectively.

Using the 'semi-Lagrangian' variables (§2) which map isopycnal surfaces into (fixed) horizontal planes, we derived (§3) the asymptotic system (3.9) which governs the dynamics of flows with horizontal spatial scale being of the order of, or larger than R_0 (defined by (1.3)). It was demonstrated (§4) that the equations derived are separable, i.e. admit a solution of the form

$$q(t,x,y,\xi) = h(t,x,y)\,\phi(\xi),$$

where h is governed by a two-dimensional equation and ϕ is an (almost) arbitrary function which describes the vertical structure of the flow. This substitution describes the equivalent-barotropic mode observed by Killworth (1987) in his analysis of the FRAM data. This and other results on the equivalent-barotropic mode in large-amplitude geostrophic flows are summarized in table 3, which suggests that, in some cases, the equivalent-barotropic mode might manifest itself more clearly, if the velocity field is represented in terms of (x, y, ρ) (ρ is the density of the fluid), instead of (x, y, z).

Within the framework of the asymptotic system derived, three sufficient conditions (5.8) of stability were derived (§5), which restrict the slope of the interface between the stratified and non-stratified layers. The results obtained were applied to the subarctic and subtropical frontal currents in the Northern Pacific in §6. The latter is found to be stable, the former is found to be unstable; but the growth rate of the instability is very small (the time of strongest growth is 20.8 months). Such a slow instability can manifest itself only if a resonance occurs between unstable oscillations of the frontal flow and the annual or biennial variability of the ocean.

It should be noted, however, that the asymptotic system (3.9) describes only *long* disturbances, i.e. those that satisfy condition (7.5*a*). The medium and short disturbances (7.5*b*, *c*) are discussed in §7, but we managed to obtain only a quantitative estimate for their stability, which does not take into account horizontal shear of the mean flow.

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